

Math 5C Discussion Problems 4

Laplace's Equation

1. Starting from scratch, use separation of variables to solve the following problem. Define $f(y) = \sin y$ and let S be the square $0 \leq x, y \leq \pi$ in \mathbb{R}^2 . Find a function u which is harmonic inside S , equal to $f(y)$ on the right edge of the square, and equal to zero on the other three sides.
2. Freely using the formulas we derived, redo the previous problem assuming:
 - (a) $f(y) = 1$.
 - (b) $f(y) = y$.
 - (c) $f(y) = \sin y$.
3. Assume R is the unbounded region in first quadrant in \mathbb{R}^2 given by $0 \leq y \leq \pi$ and that u is harmonic on R . Suppose that $u(x, y) \rightarrow 0$ as $x \rightarrow \infty$, $u(0, y) = 1$ for all y , and that $u(x, 0) = u(x, \pi) = 0$ for all x . Use separation of variables to find a formula for u .

Integration Revisted

1. Evaluate the following. Consider using series.

(a) $\int_0^{\infty} \ln(1 - e^{-x}) dx$

(b) $\int_0^{\infty} x[\ln(e^x - 1) - x] dx$

(c) $\int_0^1 \frac{\ln x}{1-x} dx$

(d) $\int_0^1 \frac{\ln x}{1-x^2} dx$

(e) $\int_0^1 x^2 \frac{\ln x}{1-x} dx$

(f) $\int_0^1 \ln x \ln(1-x) dx$

(g) $\int_0^{\infty} \frac{x}{e^x - 1} dx$

2. Evaluate the following. Try substitutions to relate to known integrals.

(a) $\int_0^{\infty} x^{-1/2} e^{-x} dx$

(b) $\int_0^{\infty} e^{-(x^2+x+1)} dx$

(c) $\int_0^{\infty} x e^{-x^4} dx$

3. Evaluate the following. Consider double integrals.

(a) $\int_0^{\infty} \frac{e^{-x} - e^{-2x}}{x} dx$

(b) $\int_0^{\infty} \frac{\arctan \pi x - \arctan x}{x} dx$

4. Evaluate the following. Consider differentiation with respect to a .

(a) Use $I(a) = \int_0^{2\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta$ to find $\int_0^{2\pi} e^{\cos \theta} \cos(\sin \theta) d\theta$.

(b) Use $I(a) = \int_0^{\infty} e^{-ax} \frac{\sin x}{x} dx$ to find $\int_0^{\infty} \frac{\sin x}{x} dx$.

(c) Use $I(a) = \int_0^1 \frac{x^a - 1}{\ln x} dx$ to find $\int_0^1 \frac{x-1}{\ln x} dx$.

5. Use clever substitutions to evaluate the following.

(a) $\int_0^{\pi/2} \ln \tan x dx$ (try $u = \pi/2 - x$)

(b) $\int_0^{\infty} \left(x + \frac{1}{x}\right)^{-2} \frac{\ln x}{x} dx$ (try $u = 1/x$)

Silly Logic Problems

Want more practice for those inevitable bonus questions?

1. A king of a distant land tests his criminals before jailing them. He presents each one with a collection of identical doors, one of which leads to freedom, while the others lead to imprisonment. Each door is labelled with a sign, upon which is written a single statement. This statement, however, could be either true or false.
 - (a) The first prisoner encounters two doors; the king says that the statements are either both true or both false. The doors read:
 - i. Freedom is behind one of these doors.
 - ii. Prison is behind the other door.Which should the prisoner pick?
 - (b) The second prisoner encounters two doors; the king says that the statements are either both true or both false. The doors read:
 - i. Either prison is behind this door or freedom is behind the other door.
 - ii. Freedom is behind the other door.Which should the prisoner pick?
 - (c) The third prisoner encounters three doors; the king says that at most one statement is true. The doors read:
 - i. Prison is behind this door.
 - ii. Freedom is behind this door.
 - iii. Prison is behind door 2.Which should the prisoner pick?
2. (Multiple choice question) The following is a list of consistent statements, each either true or false. What is the answer to this question?
 - (a) A is the answer if and only if C is false.
 - (b) B is the answer if and only if B is false.
 - (c) Both D and E are true.
 - (d) If C is false then D is false.
 - (e) Both A and B are false.
3. Of the following 100 (logically consistent) statements, which are true and which are false?
 - The next statement is false.
 - The next statement is false.
 - \vdots
 - The next statement is false.
 - The number of false statements is even.
4. There are three doors, labelled A, B, and C. There are three signs that are to be placed on the doors; in no particular order, the signs read:
 - The statement on door A is false.
 - The statement on door B is false.
 - The statement on door C is false.

If the signs are randomly placed on the doors (one sign per door), what is the probability that the resulting set of statements is logically consistent?